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Application of Chance-Constrained Programming to Solution of the So-called "Savings and Loan Association" Type of Problem

by
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McLEAN, VIRGINIA

FOREWORD

This is the third in a series of papers that deal with the problem of finding the optimal decision rules for n-period chance-constrained programming models. The first two papers in this series are entitled "Optimal Decision Rules for the E Model of Chance-Constrained Programming" and "Optimal Decision Rules for the Triangular E Model of Chance-Constrained Programming."

This paper shows how the mathematical results developed in the above-mentioned papers can be successfully applied to analysis of problems in planning under uncertainty. Such a problem has been formulated in the literature as the "Savings and Loan Problem" although the theory has relevance in many other situations, some directly concerned with military problems. The same model could be used as an aid to decision making in any situation where decisions must be made periodically over an n-period horizon. This includes such problems as planning research and development projects. The particular instance chosen to illustrate the theory was selected because earlier work has been done on this problem, and it is desirable to be able to compare the results contained here with those previously obtained by other authors.

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Nicholas M. Smith
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Application of Chance-Constrained Programming to Solution of the So-called "Savings and Loan Association" Type of Problem

ABSTRACT

This paper contains an application of chance-constrained programming to a problem in financial planning. In particular the problem is one of planning for liquidity in a savings and loan association first discussed by Charnes and Thore. The optimal rules for this problem are found and compared with the optimal linear rules given by Charnes and Thore. The discontinuous nature of the optimal rules is discussed from economic and control theory viewpoints.

Introduction

This paper gives an application of the method of chance-constrained programming to a problem in financial planning. Such problems are particularly well suited to analysis by chance-constrained programming because they deal with questions of planning in the face of an uncertain future and are such that the chance elements enter into both the objective function and the constraints. These are precisely the kinds of problems that chance-constrained programming was designed to handle.

In most of the work done thus far in chance-constrained programming, problems have been solved by converting them into equivalent deterministic and, in general, nonlinear problems. This transformation has been accomplished chiefly by restricting the class of admissible decision rules to the class of linear decision rules.[1-3]

Even purely deterministic models have proved useful in the analysis of problems of financial budgeting. Specifically, the work of Charnes, Cooper, and Miller[4] on the problem of the costing of funds in a simple warehouse model is mentioned. There in addition to the usual set of constraints in a warehouse model, is added a simple deterministic liquidity constraint requiring that the planned accumulation of cash be greater than or equal to the difference between the minimum cash balance considered acceptable and the initial cash holdings.

The specific problem presented here is a two-period problem of planning for liquidity in a savings and loan association. The model used was first discussed by Charnes and Thore.[5] Their assumption was that the admissible class of decision rules was the class of linear rules. The deterministic equivalent was found and the resulting nonlinear programming problem was solved to get the optimal decision rules in terms of the parameters of the model.

The results presented in this paper differ from these in one extremely significant way. It is no longer assumed that the class of decision rules under consideration is linear. In particular, the decision rules will be arbitrary functions of the random observations and decision rules of previous periods, subject only to certain regularity conditions that permit the use of the isoperimetric theory of the calculus of variations. Thus the admissible class of decision rules in this formulation is much larger than the class of only linear rules.

In Charnes and Kirby[6] necessary conditions were developed for decision rules, restricted only to the class described above, to be optimal for the type of problem to be considered here. Most of the mathematical results in this paper will be based on the results contained in this paper.

It is shown that under certain circumstances the optimal decision rule in this class of feasible rules is, in fact, the optimal linear rule. The result is of particular importance because it shows that good reason exists, other than the fact that it is mathematically more manageable, for limiting oneself to finding the optimal linear decision rule.

It will also be shown that in all other cases (i.e., where the rule is not linear) the optimal rule is discontinuous. This result may seem to be highly surprising at first but by economic arguments it will be shown that it does, in fact, appear to be reasonable.

Description of the Model

To provide a background for subsequent mathematical treatment of the problem a brief description of the model with some remarks about its institutional setting will be given (Ref 5, pp 5-22).

The problem is planning for liquidity in a savings and loan association. There are several reasons why the association needs to provide voluntarily for liquidity. These include the fact that the association must be ready to pay its savers the money that belongs to them on request. In other words the association must be prepared to handle withdrawals from its total savings capital.

At the same time a continuous flux exists in the mortgage-loan portfolio of an association owing to people requesting and receiving new loans, while a stream of repayments from outstanding mortgage loans brings funds back to the association. Hence in any period the association must have cash available to meet the excess of new loans over the stream of repayments.

These liquidity needs are usually met by the inflow of new savings and of mortgage-loan repayments. There are, however, other sources such as the stock liquidity, which is available by selling the US government securities contained in the association's stock portfolio. Cash can also be obtained by borrowing from both the Federal Home Loan Bank and commercial banks.

There are also legal restrictions on the amount of liquid assets that the association must hold. At the present time the required minimum ratio of cash plus US securities to savings capital is 7 percent. Thus, if an association wants to hold liquid assets, to be available in case of liquidity needs, it must hold liquid funds above this 7 percent requirement.

The problem can then be expressed as follows: Given the different needs for liquidity of a savings and loan association and given the different sources of liquidity available, how should the association choose between these alternative sources of funds in order to provide for the given needs of liquidity?

In the very simple model discussed in this paper the essential feature is that the association holds only two types of assets: cash and loans. The association's liquid assets, cash, yield no earnings at all, and the illiquid assets, loans, yield earnings through the interest paid on them although they are perfectly illiquid. Thus the association is faced with a clear-cut choice between liquidity and profitability.

Furthermore it is assumed that the making of new loans is a decision variable that is completely under the control of the association; i.e., there is perfect competition in the loan market so that the association can make any number of loans that it wishes on current standard terms.

The stochastic part of the model arises from the random variations of the savings capital of the association. The increase in withdrawable savings capital during period t will be treated as a random variable. This random variable can assume both positive and negative values. If it is positive, then more money has been deposited than withdrawn during that period, whereas if the random variable is negative the converse is true.

As we stated earlier the association needs free cash above the legal liquidity requirement to enable it to make new loans and to provide for savers who want to withdraw part of their investment. Although it is true that the association could borrow to obtain the cash it needs, there are certain chance-constrained limits on the frequency of such borrowing. Thus the problem is one of planning for liquidity in the face of the uncertain variations in savings capital, subject to the restriction that the association does not want to borrow too often.

To describe the model more rigorously the following notation is introduced:

Let M = total cash held by the association

L = mortgage loans

S = the legal liquidity minimum

S_t = total withdrawable savings capital at time t

i = dividend rate

$\Delta S_t = S_t - S_{t-1}(1+i)$ = the increase in total withdrawable savings capital during period t above dividends credited to accounts (i.e., the net increase in savings in period t)

ℓ_t = the new mortgage loans obtained by the association in period t

M_t = cash holdings at the end of period t

a_t = annuities received during period t

The assumption is that the frequency by which prepayments of loans and complete loan payoffs occur in the total loan portfolio does not change over time. Assuming further that the association expects to charge a constant rate of interest on its loans over time, the total amount of interest the loan ℓ_t will bring into the association over the life of the loan is $R\ell_t$, where R is the interest rate on loans.

Hence the entire interest earned by loans made during periods $t = 1, \ldots, T$ is

$$\sum_{t=1}^{T} R\ell_t$$

It is assumed that the objective of the association is to maximize the expected interest accrued by ℓ_t, $t = 1, \ldots, T$, i.e., it is desired to maximize $E(\Sigma_{t=1}^{T} R\ell_t)$.

It is supposed that a known constant λ exists such that the association is required by law to keep cash balances amounting to at least 100λ percent of the total savings capital S. Moreover, borrowing is not allowed when the association holds cash above the legal minimum, i.e., when $M_t \geq \lambda S_t$. It is also assumed that the association cannot and does not want to borrow too often and therefore needs to check the frequency of its borrowing. This leads to the liquidity constraint

$$P(M_t \geq \lambda S_t) \geq \alpha_t \qquad t = 1, \ldots, T, \tag{1}$$

where α_t is some preassigned probability.

The admissible class of decision rules—distribution of the random variables ΔS_t will be discussed in a moment—is such that it allows ℓ_t to be negative. This shall be interpreted as term borrowing on the same length of time and with the same interest charge as given by the average term loan. It is assumed

that such long-term borrowing cannot occur too frequently, therefore the constraint

$$P(\ell_t \geq 0) \geq \beta_t \qquad t = 1, \ldots, T, \tag{2}$$

where β_t is a preassigned probability, is imposed on ℓ_t.

In each period t, $t = 1, \ldots, T$, the budget constraint (balance-sheet identity) is

$$M_t + \ell_t = M_{t-1} + a_t + \Delta S_t \tag{3}$$

This says that the sum of the cash holdings at the end of period t and the new loans made during period t must equal the sum of the initial cash held at the start of period t, the annuities received during period t, and the net inflow of new savings during period t above dividends credited to accounts. In brief, the left side of Eq 3 gives the use of funds in period t and the right side gives the total loanable funds that become available during period t.

Thus the problem is to

maximize

$$E\left(\sum_{t=1}^{T} R\ell_t\right) \tag{4}$$

subject to Eqs 1, 2, and 3.

Using the data given in Charnes and Thore[5] the following expressions were obtained for the annuities a_1 and a_2, in the two-period problem $T = 2$

$a_1 \;=\; 13{,}104$

$a_2 \;=\; 11{,}762 + 0.173\,\ell_1$

in millions of dollars.

Inserting these expressions into the budget constraint Eq 3 the following expressions for M_1 and M_2 are derived:

$M_1 \;=\; -\ell_1 + \Delta S_1 + 13{,}104 + M_0$

$M_2 \;=\; -0.827\ell_1 - \ell_2 + \Delta S_1 + \Delta S_2 + 24{,}866 + M_0$.

By substituting these expressions into the liquidity constraints Eq 1 the problem can be mathematically expressed as

maximize

$$E(R\ell_1 + R\ell_2)$$

subject to

$$\begin{aligned}
P(-\ell_1 + \Delta S_1 + 13{,}104 + M_0 \geq \lambda S_1) &\geq \alpha_1, \\
P(-0.827\,\ell_1 - \ell_2 + \Delta S_1 + \Delta S_2 + 24{,}866 + M_0 \geq \lambda S_2) &\geq \alpha_2, \\
P(\ell_1 \geq 0) &\geq \beta_1, \\
P(\ell_2 \geq 0) &\geq \beta_2.
\end{aligned} \tag{5}$$

It is known from the definition of ΔS_t that $S_t = \Delta S_t + S_{t-1}(1+i)$. Therefore, $S_1 = \Delta S_1 + (1+i)S_0$ and $S_2 = \Delta S_2 + (1+i)S_1 = \Delta S_2 + (1+i)\Delta S_1 + (1+i)^2 S_0$, thus these expressions can be substituted into Eq 5. Moreover, since $R > 0$ and appears in the objective function as the coefficient of both ℓ_1 and ℓ_2, it can be dropped from the problem. Hence Eq 5 becomes

maximize

$$E(\ell_1 + \ell_2)$$

subject to

$$
\begin{aligned}
P[-\ell_1 + (1-\lambda)\Delta S_1 \geq \lambda(1+r)S_0 - M_0 - 13{,}104] &\geq \alpha_1, \\
P[-0.827\ell_1 - \ell_2 + [1-\lambda(1+r)]\Delta S_1 + (1-\lambda)\Delta S_2 \geq \lambda(1+r)^2 S_0 - M_0 - 24{,}866] &\geq \alpha_2, \\
P(\ell_1 \geq 0) &\geq \beta_1, \\
P(\ell_2 \geq 0) &\geq \beta_2.
\end{aligned}
\tag{6}
$$

Finally defining

$$
\begin{cases}
K_1 = \lambda(1+r)S_0 - M_0 - 13{,}104, \\
K_2 = \lambda(1+r)^2 S_0 - M_0 - 24{,}866,
\end{cases}
\tag{7}
$$

and so Eq 6 can be written as
maximize

$$E(\ell_1 + \ell_2)$$

subject to

$$
\begin{aligned}
P[(-\ell_1 + (1-\lambda)\Delta S_1 \geq K_1)] &\geq \alpha_1, \\
P[-0.82\ \ldots\ {}_2 + [1-\lambda(1+r)]\Delta S_1 + (1-\lambda)\Delta S_2 \geq K_2] &\geq \alpha_2, \\
P(\ell_1 \geq 0) &\geq \beta_1, \\
P(\ell_2 \geq 0) &\geq \beta_2.
\end{aligned}
\tag{8}
$$

In the development of the model thus far nothing has been said about what distributions will be assumed for the random variables ΔS_1 and ΔS_2. In Charnes and Thore[5] it was assumed that ΔS_t, $t = 1, 2$, were independent random variables each being normally distributed with mean E_t and standard deviation δ_t.

Because the following mathematical results are applicable for a fairly large class of random variables consideration at this time will not be limited to only normal random variables. Instead it is assumed that $\Delta S_1, \Delta S_2$ are independent, continuous random variables with frequency functions $\widetilde{f}_t(\Delta S_t)$, $t = 1, 2$, and distribution functions $\widetilde{F}_t(\Delta S_t)$, $t = 1, 2$, respectively. It is assumed also that either

$$\frac{d\widetilde{f}_2[\widetilde{F}_2^{-1}(1)]}{d(\Delta S_2)} = 0, \text{ or } \widetilde{F}_2^{-1}(1) = +\infty$$

and either

$$\widetilde{F}_2^{-1}(0) = -\infty$$

or

$$\widetilde{f}_2\left[\widetilde{F}_2^{-1}(D)\right] \geq \widetilde{f}_2\left[\widetilde{F}_2^{-1}(0)\right] \text{ for all } D \text{ in } (0,1)$$

or

$$[1 - \lambda(1+r)]\, Z - 0.827\ell_1 - K_2 + (1-\lambda)\widetilde{F}_2^{-1}(0) \geq 0$$

implies

$$Z \geq \widetilde{F}_1^{-1}(0) \text{ for all } \ell_1 \geq 0.$$

It is emphasized that conditions (a) and (b) do not restrict the distribution of ΔS_1. Hence ΔS_1 can be any continuous random variable. Since the 0 and 1 fractile points of a normal random variable are $-\infty$ and $+\infty$ respectively, it can be seen that ΔS_1 and ΔS_2, being normally distributed, are admissible random variables.

Mathematical Treatment

The problem now is solving Eq 8 for the optimal decision rules ℓ_1' and ℓ_2'. As is customary in n period problems in chance-constrained programming it is desired that the jth period decision rule be an explicit function of the random variables whose values will have been observed at the time the jth period decision rule is put into effect, but it is not a function of the random variables of the jth or future period.[2,6,7] Thus it is required that ℓ_2 be such that $\ell_2 = \ell_2(\Delta S_1)$, i.e., ℓ_2 is a function of ΔS_1 but not ΔS_2. This agrees with the above requirement. Since the second-period decision ℓ_2 must be made before the random variable ΔS_2 of the second period is observed, the knowledge of the observed value of ΔS_1 can be used in making the decision ℓ_2. Similarly it is required that ℓ_1 be a zero-order rule (Ref 2), since ℓ_1 is not to be an explicit function of either ΔS_1 or ΔS_2.

The probability and the expectation will be computed in Eq 8 using the joint distribution of $\Delta S_1, \Delta S_2$. Hence

$$P\{-0.827\ell_1 - \ell_2(\Delta S_1) + [1-\lambda(1+i)]\Delta S_1 + (1-\lambda)\Delta S_2 \geq K_2\}$$
$$\int_A \int \tilde{f}_1(\Delta S_1)\tilde{f}_2(\Delta S_2)\, d(\Delta S_1)\, d(\Delta S_2),$$

where A is the set of points for which

$$-0.827\ell_1 - \ell_2 + [1-\lambda(1+i)]\Delta S_1 + (1-\lambda)\Delta S_2 \geq K_2.$$

This integration over A can be written in the form

$$\int_g^h \tilde{f}_1(\Delta S_1)\left\{\int_a^{+\infty} \tilde{f}_2(\Delta S_2)\, d(\Delta S_2)\right\} d(\Delta S_1),$$

where

$$a = \frac{K_2 + 0.827\ell_1 + \ell_2(\Delta S_1) - [1-\lambda(1+i)]\Delta S_1}{(1-\lambda)}.$$

Thus the integration over A is equal to

$$1 - \int_g^h \tilde{F}_2\left\{\frac{K_2 + 0.827\ell_1 + \ell_2(\Delta S_1) - [1-\lambda(1+i)]\Delta S_1}{(1-\lambda)}\right\}\tilde{f}_1(\Delta S_1)\, d(\Delta S_1),$$

where g, h are the smallest and largest values of ΔS_1 for which $\tilde{f}_1(\Delta S_1) > 0$.*
Thus

$$P(-0.827\ell_1 - \ell_2 + [1-\lambda(1+i)]\Delta S_1 + (1-\lambda)\Delta S_2 \geq K_2) \geq \alpha_2$$

if and only if

$$\int_g^h \tilde{F}_2\left\{\frac{K_2 + 0.827\ell_1 + \ell_2(\Delta S_1) - [1-\lambda(1+i)]\Delta S_1}{1-\lambda}\right\}\tilde{f}_1(\Delta S_1)\, d(\Delta S_1) \leq 1 - \alpha_2.$$

*g, h may be $-\infty$ and $+\infty$, respectively.

Thus write Eq 8 as
maximize

$$\ell_1 + \int_g^h \ell_2(b_1)\tilde{f}_1(b_1)\,d(b_1)$$

subject to

$$\tilde{F}_1\left[\frac{\ell_1 + K_1}{1-\lambda}\right] \le 1 - \alpha_1,$$

$$\int_g^h \tilde{F}_2\left\{\frac{0.827\ell_1 + K_2 + \ell_2(b_1) - [1-\lambda(1+i)]b_1}{1-\lambda}\right\}\tilde{f}_1(b_1)\,d(b_1) \le 1 - \alpha_2, \tag{9}$$

$$P(\ell_1 \ge 0) \ge \beta_1,$$

$$P(\ell_2 \ge 0) \ge \beta_2,$$

where $b_1 = \Delta S_1$ to conform to the notation used in Charnes and Kirby.[6]

In order to solve Eq 9 assume that ℓ_1 is fixed and then proceed to find the optimal ℓ_2, namely, ℓ_2', in terms of ℓ_1 and b_1. Having found ℓ_2' put it into the objective function of Eq 9 to find $E(\ell_2')$ in terms of ℓ_1 and then solve the resulting problem for ℓ_1', the optimal value of ℓ_1.

Thus consider the problem
maximize

$$\int_g^h \ell_2(b_1)\tilde{f}_1(b_1)\,d(b_1)$$

subject to

$$\int_g^h \tilde{F}_2\left\{\frac{0.827\ell_1 + \ell_2 - [1-\lambda(1+i)]b_1 + K_2}{1-\lambda}\right\}\tilde{f}_1(b_1)\,d(b_1) \le 1 - \alpha_2,$$

$$P(\ell_2 \ge 0) \ge \beta_2, \tag{10}$$

where it is assumed that ℓ_1 is a known constant.

In order to use the results contained in Charnes and Kirby[6] one further assumption is required.

There exists a partition of the set of points for which $\tilde{f}_1(b_1) > 0$ such that for each interval $[y_j', z_j']$ in the partition the following properties hold:

(a) $\int_{y_j'}^{z_j'} \tilde{f}_1(b_1)\,d(b_1) > 0,$

(b) $\tilde{f}_1,\ \ell_2'\tilde{f}_2\left\{\frac{0.827\ell_1 + \ell_2' - [1-\lambda(1+i)]b_1 + K_2}{1-\lambda}\right\}$ are continuous in $[y_j', z_j']$,

and

(c) ℓ_2' is of constant sign in $[y_j', z_j']$.

This partition shall be referred to as the "optimal partition" of $[g, h]$.

From theorem 3 in Charnes and Kirby[6], it is known that in any interval $[y_j', z_j']$, $\ell_2'(b_1)$ is defined by one of the following four equations:

$$\ell_2'(b_1) = 0, \tag{11}$$

$$\ell_2'(b_1) = [1-\lambda(1+i)]b_1 - 0.827\ell_1 - K_2 + (1-\lambda)\tilde{F}_2^{-1}(D^-), \tag{12}$$

$$\ell_2'(b_1) = [1-\lambda(1+i)]b_1 - 0.827\ell_1 - K_2 + (1-\lambda)\tilde{F}_2^{-1}(0), \tag{13}$$

$$\ell_2'(b_1) = [1-\lambda(1+i)]b_1 - 0.827\ell_1 - K_2 + (1-\lambda)\tilde{F}^{-1}(1), \tag{14}$$

where D'' satisfies $0 < D'' < 1$ and is a constant.

From theorem 7[6] it is known that a further necessary condition that $\ell_2'(b_1)$ be defined by Eq 14 over any interval $[y_j', z_j']$ in the optimal partition is that $\tilde{F}_2^{-1}(1) < +\infty$ and satisfy

$$\frac{d\tilde{f}_2[\tilde{F}_2^{-1}(1)]}{db_2} \geq 0.$$

The admissible class of random variables violates this inequality when $\tilde{F}_2^{-1}(1) < +\infty$; hence it is concluded that $\ell_2'(b_1)$ is not given by Eq 14 for any interval in the optimal partition.

From theorem 8[6] a necessary condition that $\ell_2'(b_1)$ be given by Eq 13 over some interval $[y_j', z_j']$ in the optimal partition is that $\tilde{F}_2^{-1}(0) > -\infty$, and either

$$\tilde{f}_2[\tilde{F}_2^{-1}(0)] \geq \tilde{f}_2[\tilde{F}_2^{-1}(D'')],$$

or

$$[1-\lambda(1+i)]\bar{b}_1 - 0.827\ell_1 - K_2 + (1-\lambda)\tilde{F}_2^{-1}(0) = 0$$

for some point $\bar{b}_1$ in $[y_j', z_j']$.

Again, it can be seen that the admissible class of random variables fails to satisfy these conditions. Therefore ℓ_2' is not given by Eq 13 anywhere.

Hence it has been shown, using the results in Charnes and Kirby[6] and the assumptions concerning the distributions of b_1 and b_2, that a necessary condition that $\ell_2'(b_1)$ be optimal for Eq 10 is that there exists a partition of $[g, h]$, which is denoted by $[y_j', z_j']$, with properties (a), (b), and (c) previously described and such that in each $[y_j', z_j']$ either

$$\ell_2'(b_1) = 0,$$

or

$$\ell_2'(b_1) = [1-\lambda(1+i)]b_1 - 0.827\ell_1 - K_2 + (1-\lambda)\tilde{F}_2^{-1}(D''),$$

where D'' satisfies $0 < D'' < 1$ and D'' is a constant.

There are two possible cases, one in which the constraint $P(\ell_2' \geq 0) \geq \beta_2$ is binding and the other where it is not binding. In the latter case theorem 6 Charnes and Kirby[6] can be used to conclude that $\ell_2'(b_1)$ is not identically 0 over any interval in our optimal partition. Thus

Lemma 1: If $P[\ell_2(b_1) \geq 0] \geq \beta_2$ is not binding in Eq 10, then for $g \leq b_1 \leq h$,

$$\ell_2'(b_1) = [1-\lambda(1+i)]b_1 - 0.827\ell_1 - K_2 + (1-\lambda)\tilde{F}_2^{-1}(D'').$$

An examination of the first constraint of Eq 10 shows that it must be satisfied as an equality at the optimum. Otherwise, $\ell_2'(b_1)$ could be increased for some value of b_1, as the integrand of the constraint is a monotonic increasing function of ℓ_2, thus increasing the value of the objective function and therefore

contradicting the optimality of $\ell_2'(b_1)$. This equality in the $1 - \alpha_2$ constraint will give

$$D'' = 1 - \alpha_2 \tag{15}$$

when Lemma 1 holds.

The case in which the constraint $P\{\ell_2' > 0\} \geq \beta_2$ is binding is next considered.

Let $I = [j : \ell_2' \geq 0$ with equality on at most a set of measure 0 in $(y_j', z_j')]$.
Let $J = [j : \ell_2' \leq 0$ with equality on at most a set of measure 0 in $(y_j', z_j')]$.
Let $K = [j : \ell_2' = 0$ in $(y_j', z_j')]$.

Suppose that D'' is known. Then the problem of determining y_j', z_j', for $j \epsilon I$ and $j \epsilon J$, can be expressed as
maximize

$$\Sigma_{j\epsilon I,J}\left[(1-\lambda)\,\widetilde{F}_2^{-1}(D'') - K_2 - 0.827\ell_1\right]\int_{y_j}^{z_j}\widetilde{f}_1(b_1)\,d(b_1)$$

$$+\ \Sigma_{j\epsilon I,J}\left[1-\lambda(1+i)\right]\int_{y_j}^{z_j} b_1\,\widetilde{f}_1(b_1)\,db_1$$

subject to

$$D''\,\Sigma_{j\epsilon I,J}\int_{y_j}^{z_j}\widetilde{f}_1(b_1)db_1 - \Sigma_{j\epsilon I,J}\int_{y_j}^{z_j}\widetilde{F}_2\left\{\frac{0.827\ell_1 - [1-\lambda(1+i)]b_1 + K_2}{1-\lambda}\right\}\widetilde{f}_1(b_1)\,d(b_1)$$

$$= 1 - \alpha_2 - \int_a^h \widetilde{F}_2\left\{\frac{0.827\ell_1 - [1-\lambda(1+i)]b_1 + K_2}{1-\lambda}\right\}\widetilde{f}_1\,d(b_1),$$

$$\Sigma_{j\epsilon J}\int_{y_j}^{z_j}\widetilde{f}_1(b_1)d(b_1) = 1-\beta_2, \tag{16}$$

$$y_j \leq z_j,\ j\epsilon I \text{ and } j\epsilon J,$$

$$z_{j-1} \leq y_j,\ j\epsilon I \text{ and } j\epsilon J,$$

$$a \leq y_j,\ j\epsilon I \text{ and } j\epsilon J,$$

$$z_j \leq h,\ j\epsilon I \text{ and } j\epsilon J,$$

and

$$D'' \geq \widetilde{F}_2\left\{\frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]b_1}{1-\lambda}\right\} \text{ for all } b_1 \text{ in } [y_j, z_j] \text{ when } j\epsilon I,$$

$$D'' \leq \widetilde{F}_2\left\{\frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]b_1}{1-\lambda}\right\} \text{ for all } b_1 \text{ in } [y_j, z_j] \text{ when } j\epsilon J.$$

In Eq 16 the objective is obtained by using the expression for $\ell_2'(b_1)$ given in Eq 12 since it is only necessary to perform the integration over those intervals for which $\ell_2'(b_1)$ is not identically 0, i.e., for $j\epsilon I$ and $j\epsilon J$. The first constraint of Eq 16 corresponds to the first constraint of Eq 10, except that it is written in such a way that it involves only the intervals $[y_j, z_j]$ for which $j\epsilon I$ and $j\epsilon J$. The second constraint says the $P(\ell_2' < 0) = 1 - \beta_2$, which is true if and only if $P(\ell_2' > 0) = \beta_2$; therefore it corresponds to the second constraint of Eq 10. The remaining conditions assure that a collection of nonoverlapping intervals will be secured such that $\ell_2' \geq 0$ for $j\epsilon I$ and $\ell_2' \leq 0$ for $j\epsilon J$. The fact that $\ell_2' = 0$ on at most a set of measure 0 in any of the intervals for $j\epsilon I$ and $j \epsilon J$ is guaranteed by the fact that $\widetilde{F}_2\ [0.827\ell_1 + K_2 - [1-\lambda(1+i)]b_1]/ 1-\lambda$ is a

strictly monotonic decreasing function of b_1 over the sets where $\tilde{f}_1(b_1) > 0$, as $[1-\lambda(1+i)] > 0$.

Using this strict monotonicity the last two constraints of Eq 16 can be replaced by

$$D^* \leq \tilde{F}_2 \left\{ \frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]y_j}{1-\lambda} \right\} \text{ for all } j \in I,$$

and

$$D^* \leq \tilde{F}_2 \left\{ \frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]z_j}{1-\lambda} \right\} \text{ for all } j \in J.$$

Equation 16 will now be solved using the Lagrange multiplier technique to derive necessary conditions for y_j, z_j to be optimal for $j \in I$, $j \in J$. To do this begin by forming the Lagrangian function

$$\begin{aligned}
L = {} & [(1-\lambda)\tilde{F}_2^{-1}(D^*) - K_2 - 0.827\ell_1] \Sigma_{j\in I,J} \int_{y_j}^{z_j} \tilde{f}_1(b_1)\, d(b_1) \\
& + [1-\lambda(1+i)] \Sigma_{j\in I,J} \int_{y_j}^{z_j} b_1 \tilde{f}_1(b_1)\, d(b_1) \\
& + \Omega \left(D^* \Sigma_{j\in I,J}[\tilde{F}_1(z_j) - \tilde{F}_1(y_j)] - \Sigma_{j\in I,J} \int_{y_j}^{z_j} \tilde{F}_2 \left\{ \frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]b_1}{1-\lambda} \right\} \tilde{f}_1(b_1)\, d(b_1) - (1-\alpha_2) - Q \right) \\
& + \Delta \left\{ \Sigma_{j\in J} [\tilde{F}_1(z_j) - \tilde{F}_1(y_j)] - (1-\beta_2) \right\} \\
& + \Sigma_{j\in I,J} \eta_j (y_j - z_j + T_j^2) + \Sigma_{j\in I,J} \xi_j (y_j - z_{j-1} - R_j^2) \\
& + \Sigma_{j\in I,J} \delta_j (y_j - g - Q_j^2) + \Sigma_{j\in I,J} \phi_j (h - z_j - P_j^2) \\
& + \Sigma_{j\in I} \lambda_j \left(D^* - \tilde{F}_2 \left\{ \frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]y_j}{1-\lambda} \right\} - S_j^2 \right) \\
& + \Sigma_{j\in J} \theta_j \left(D^* - \tilde{F}_2 \left\{ \frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]z_j}{1-\lambda} \right\} + W_j^2 \right)
\end{aligned}$$

where Δ, Ω, η_j, ξ_j, δ_j, ϕ_j, λ_j, θ_j, are Lagrange multipliers and

$$Q = \int_g^h \tilde{F}_2 \left\{ \frac{0.827\ell_1 - [1-\lambda(1+i)]b_1 + K_2}{1-\lambda} \right\} \tilde{f}_1(b_1)\, d(b_1).$$

The following equations and inequalities provide some necessary conditions that y_j', z_j' maximize L and hence solve Eq 16.

Let $j \in I$

Then $\partial L / \partial z_j = 0$ implies

$$z_j = \frac{\eta_j + \xi_{j+1} + \phi_j}{[1-\lambda(1+i)]\tilde{f}_1(z_j)} - \left[\frac{(1-\lambda)\tilde{F}_2^{-1}(D^*) - K_2 - 0.827\ell_1}{1-\lambda(1+i)} \right] - \frac{\left(\Omega D^* - \Omega\tilde{F}_2 \left\{ \frac{0.827\ell_1 - [1-\lambda(1+i)]z_j + K_2}{1-\lambda} \right\} \right)}{1-\lambda(1+i)} \tag{17}$$

and $\partial L/\partial y_j = 0$ implies

$$y_j = \frac{\eta_j + \xi_j + \delta_j}{[1-\lambda(1+i)]\widetilde{f}_1(y_j)} - \frac{[1-\lambda)\widetilde{F}_2^{-1}(D^{\sim}) - K_2 - 0.827\ell_1]}{1-\lambda(1+i)}$$

$$- \frac{\left(\Omega D^{\sim} - \Omega\widetilde{F}_2\left\{\frac{0.827\ell_1 - [1-\lambda(1+i)]y_j + K_2}{1-\lambda}\right\}\right)}{1-\lambda(1+i)} \tag{18}$$

$$+ \frac{\lambda_j \widetilde{f}_2\left\{\frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]y_j}{1-\lambda}\right\}}{[1-\lambda]\widetilde{f}_2(y_j)} .$$

Let $j \epsilon J$

Then $\partial L/\partial z_j = 0$ implies

$$z_j = \frac{\eta_j + \xi_{j+1} + \phi_j}{[1-\lambda(1+i)]\widetilde{f}_1(z_j)} - \frac{[(1-\lambda)\widetilde{F}_2^{-1}(D^{\sim}) - K_2 - 0.827\ell_1]}{1-\lambda(1+i)}$$

$$- \frac{\left(\Omega D^{\sim} - \Omega\widetilde{F}_2\left\{\frac{0.827\ell_1 - [1-\lambda(1+i)]z_j + K_2}{1-\lambda}\right\}\right)}{1-\lambda(1+i)} \tag{19}$$

$$- \frac{\Lambda}{1-\lambda(1+i)} - \frac{\left(\theta_j \widetilde{f}_2\left\{\frac{0.827\ell_1 - [1-\lambda(1+i)]z_j + K_2}{1-\lambda}\right\}\right)}{[1-\lambda]\widetilde{f}_2(z_j)}$$

and $\partial L/\partial y_j = 0$ implies

$$y_j = \frac{\eta_j + \xi_j + \delta_j}{[1-\lambda(1+i)]\widetilde{f}_1(y_j)} - \frac{[(1-\lambda)\widetilde{F}_2^{-1}(D^{\sim}) - K_2 - 0.827\ell_1]}{1-\lambda(1+i)}$$

$$- \frac{\left(\Omega D^{\sim} - \Omega\widetilde{F}_2\left\{\frac{0.827\ell_1 - [1-\lambda(1+i)]y_j + K_2}{1-\lambda}\right\}\right)}{1-\lambda(1+i)} \tag{20}$$

$$- \frac{\Lambda}{1-\lambda(1+i)} .$$

$$\frac{\partial L}{\partial T_j} = 0 \cdot \eta_j T_j = 0, \quad j \epsilon I \text{ and } j \epsilon J, \tag{21}$$

$$\frac{\partial L}{\partial R_j} = 0 \cdot -\xi_j R_j = 0, \quad j \epsilon I \text{ and } j \epsilon J. \tag{22}$$

$$\frac{\partial^2 L}{\partial R_j^2} \leq 0 \cdot \xi_j \geq 0, \quad j \epsilon I \text{ and } j \epsilon J, \tag{23}$$

$$\frac{\partial L}{\partial S_j} = 0 \cdot -\lambda_j S_j = 0, \quad j \epsilon I, \tag{24}$$

$$\frac{\partial^2 L}{\partial S_j^2} \leq 0 \cdot \lambda_j \geq 0, \quad j \epsilon I, \tag{25}$$

$$\frac{\partial^2 L}{\partial W_j} = 0 \cdot \theta_j W_j = 0, \qquad j \epsilon J, \tag{26}$$

$$\frac{\partial^2 L}{\partial W_j^2} \leq 0 \cdot \theta_j \leq 0, \qquad j \epsilon J. \tag{27}$$

From the expression Eq 12 for ℓ_2' in any interval $[y_j, z_j]$ for which $j \epsilon I$, or $j \epsilon J$, it is seen that ℓ_2' is a straight line of positive slope as $[1-\lambda(1+i)] > 0$. Therefore if at some point,say $\overline{b}_1$, $\ell_2'(\overline{b}_1) > 0$ then $\ell_2'(\overline{b}_1) \geq 0$ for all $b_1 \geq \overline{b}_1$. This conclusion follows directly from the geometry of the situation.

Let m be the smallest value of j for which $\ell_2' \geq 0$ almost everywhere in $[y_j, z_j]$. Then $\ell_2' \geq 0$ in all intervals $[y_j, z_j]$ for which $j \geq m$.

Similarly $\ell_2' \leq 0$ in all intervals $[y_j, z_j]$ for which $j < m$.

Next, it is observed from Eq 21 that $\eta_j \neq 0 \rightarrow T_j = 0 \rightarrow y_j = z_j$ from the equation $\partial L/\partial \eta_j = 0$. But $y_j = z_j$ implies that Eq 16 could just as well be solved by dropping the value of j from the problem. So the assumption now is that $\eta_j = 0$ for all j.

Moreover from Eq 22 $\xi_{j+1} \neq 0 \rightarrow R_{j+1} = 0 \rightarrow y_{j+1} = z_j$ from the equation $\partial L/\partial \xi_{j+1} = 0$. But when $j \geq m$ it is known that $\ell_2' \neq 0 \rightarrow \ell_2' > 0$, so that $y_{j+1} = z_j$ means that in the intervals $[y_j, z_j]$ and $[y_{j+1}, z_{j+1}]$, ℓ_2' is defined by $\ell_2' = [1-\lambda(1+i)]\, b_1 - 0.827\, \ell_1 - K_2 + (1-\lambda)\tilde{F}_2^{-1}(D'')$. If this is the case, then the intervals $[y_j, z_j]$, $[y_{j+1}, z_{j+1}]$ can be written as one continuous interval $[y_j, z_{j+1}]$. Thus the indexes could be renumbered dropping one interval from the problem. It is assumed that $\xi_{j+1} = 0, j \geq m$.

Similarly it is assumed that $\xi_{j+1} = 0$ for $j \leq m-2$. Therefore the only ξ_j that can be nonzero is ξ_m.

Using an analogous argument the only δ_j that can be nonzero is δ_k, and the only ϕ_j that can be nonzero is ϕ_ℓ, where ℓ is such that $z_j \leq z_\ell$, $j \epsilon I$, $j \epsilon J$, and k is such that $y_j \geq y_k$, $j \epsilon I$, $j \epsilon J$.

Now if $j \epsilon I$ then $\xi_{j+1} = 0$, and if $\phi_j = 0$ also, then Eq 17 implies that there is only one z_j for $j \epsilon I$. This follows from the fact that the solution of the equation

$$z_j = -\frac{[(1-\lambda)\tilde{F}_2^{-1}(D'') - K_2 - 0.827\,\ell_1]}{1-\lambda(1+i)} - \frac{\left(\Omega D'' - \Omega \tilde{F}_2 \left\{\dfrac{0.827\,\ell_1 + K_2 - [1-\lambda(1+i)]\, z_j}{1-\lambda}\right\}\right)}{1-\lambda(1+i)}$$

is unique, as $\tilde{F}_2$ is a monotonic function of z_j and it is known that Eq 17 must hold for all $j \epsilon I$.

If $\phi_j \neq 0$ then $j = \ell$ from the work above. Hence using $\partial L/\partial P_\ell = 0$ it is found that $P_\ell = 0 \rightarrow z_\ell = h$ from the equation $\partial L/\partial \phi_\ell = 0$, so again z_j is uniquely determined. Hence, there exists at most one interval $[y_j, z_j]$ for $j \epsilon I$.

Similarly by using Eq 20 and our definition of δ_k, it is shown that there exists at most one interval $[y_j, z_j]$ for $j \epsilon J$.

Using the fact that ℓ_2' is a monotonic increasing function of b_1 in $[y_j', z_j']$, $j \epsilon I$ and $j \epsilon J$, and that I, J contain at most one index j, it can be seen that the constraints restricting $\ell_2' \geq 0$, $j \epsilon I$, and $\ell_2' \leq 0$, $j \epsilon J$, can, in fact, be replaced by $\ell_2' > 0$ when $j = m$, $\ell_2' < 0$ when $j = m-1$, and $\ell_2' = 0$ for $j \neq m-1, m$. Note

also that it is necessary to have $\delta_m = 0$, as $\delta_m \neq 0 \rightarrow y_m = g \rightarrow J$ is empty, which is impossible since $\beta_2 \neq 0$.

Since Eq 16 is being solved for y_j', z_j' for $j \epsilon I$ and $j \epsilon J$, it is now only necessary to consider the cases in which $j = m$ and $j = m - 1$.

Let $j = m$ and suppose $z_m < h$ so that $P_m \neq 0$. Then $\delta_m = 0$ from the equation $\partial L/\partial P_m = 0$, and the following possible cases exist.

(a) $\xi_m = 0$ and $\lambda_m = 0$. Then Eq 17 and Eq 18 imply $y_m = z_m$; i.e., I is empty.

(b) $\xi_m = 0$ and $\lambda_m \neq 0$. Then $S_m = 0$ from Eq 24; hence, from $\partial L/\partial\lambda_m = 0$, there is

$$D'' = \tilde{F}_2\left\{\frac{0.827 f_1 + K_2 - [1-\lambda(1+\eta)]y_m}{1-\lambda}\right\}$$

or

$$y_m = \frac{[-(1-\lambda)\tilde{F}_2^{-1}(D'') + 0.827 f_1 + K_2]}{1-\lambda(1+\eta)}. \tag{28}$$

Putting this into Eq 18 with $j = m$, using the fact that $\delta_m = 0$, the result is

$$\frac{-\Omega}{1-\lambda(1+\eta)}\left(D'' - \tilde{F}_2\left\{\frac{0.827 f_1 + K_2 - [1-\lambda(1+\eta)]\dfrac{[-(1-\lambda)\tilde{F}_2^{-1}(D'') + 0.827 f_1 + K_2]}{1-\lambda(1+\eta)}}{1-\lambda}\right\}\right)$$
$$+ \frac{\lambda_m \tilde{f}_2\left\{\dfrac{0.827 f_1 + K_2 - [1-\lambda(1+\eta)]y_m}{1-\lambda}\right\}}{(1-\lambda)\tilde{f}_2(y_m)} = 0,$$

or

$$\frac{-\Omega D'' + \Omega D''}{1-\lambda(1+\eta)} + \frac{\lambda_m \tilde{f}_2[\tilde{F}_2^{-1}(D'')]}{[1-\lambda]\tilde{f}_2(y_m)} = 0.$$

Hence, the result is

$$\frac{\lambda_m \tilde{f}_2[\tilde{F}_2^{-1}(D'')]}{(1-\lambda)\tilde{f}_2(y_m)} = 0. \tag{29}$$

But $\tilde{f}_2[\tilde{F}_2^{-1}(D'')] \neq 0$ as $0 < D'' < 1$, therefore Eq 29 implies $\lambda_m = 0$, which is a contradiction. Therefore $\delta_m = 0$, $\xi_m \neq 0$ and $\lambda_m \neq 0$ is impossible.

(c) $\xi_m \neq 0$ and $\lambda_m = 0$. Then $R_m = 0 \rightarrow y_m = z_{m-1}$. But $\ell_2'(y_m) \geq 0$ and $\ell_2(z_{m-1}) \leq 0$ from the definition of m, hence $\ell_2'(y_m) = 0$. This implies that Eq 28 must again hold, therefore, by inserting Eq 28 into Eq 18 the following equation must be true:

$$\frac{\xi_m}{[1-\lambda(1+\eta)]f_1(y_m)} = 0, \text{ or } \xi_m = 0,$$

which is a contradiction. Therefore

$$\delta_m = 0, \ \xi_m \neq 0 \text{ and } \lambda_m = 0$$

is impossible.

(d) $\xi_m \neq 0$ and $\lambda_m \neq 0$. Again Eq 28 holds; hence

$$\frac{\lambda_m \tilde{f}_2[\tilde{F}_2^{-1}(D'')]}{(1-\lambda)\tilde{f}_2(y_m)} + \frac{\xi_m}{[1-\lambda(1+i)]\tilde{f}_1(y_m)} = 0.$$

Therefore $\lambda_m = 0$ and $\xi_m = 0$ as by Eqs 23 and 25 $\lambda_m > 0$, $\xi_m \geq 0$, and it is known that $(1-\lambda)$, $[1-\lambda(1+i)]$, $\tilde{f}_1(y_m)$, $\tilde{f}_2(y_m)$, and $\tilde{f}_2[\tilde{F}_2^{-1}(D'')]$ are all positive. Thus a contradiction exists.

Therefore

$$\begin{cases} \delta_m = 0 \Rightarrow I \text{ is empty}, \\ \delta_m \neq 0 \Rightarrow z_m = h. \end{cases} \tag{30}$$

Using an analogous development by assuming that $\delta_{m-1} = 0$ similar contradictions are derived and it is concluded that

$$\begin{cases} \delta_{m-1} = 0 \Rightarrow J \text{ is empty}, \\ \delta_{m-1} \neq 0 \Rightarrow y_{m-1} = g. \end{cases} \tag{31}$$

It has already been shown that at most only one interval $[y_j, z_j]$ exists in I or J. So using the constraint that $\Sigma_{j \in J} [\tilde{F}_1(z_j) - \tilde{F}_1(y_j)] = 1 - \beta_2$, it is seen that J cannot be empty; hence $y'_{m-1} = g$, $z'_{m-1} = \tilde{F}_1^{-1}(1-\beta_2)$.

Therefore

$$\ell_2'(b_1) = [1-\lambda(1+i)]b_1 - 0.827\ell_1' - K_2 + (1+\lambda)\tilde{F}_2^{-1}(D'')$$
$$\text{for } g \leq b_1 \leq \tilde{F}_1^{-1}(1-\beta_2). \tag{32}$$

Moreover using Eqs 30 and 32 Eq 9 can be written as the following nonlinear problem in the three variables D^2, y_m, and ℓ_1:

maximize

$$\ell_1 + [(1-\lambda)\tilde{F}_2^{-1}(D) - K_2 - 0.827\ell_1][1 - \tilde{F}_1(y_m)]$$
$$+ [1-\lambda(1+i)] \int_{y_m}^{h} b_1 \tilde{f}_1(b_1)\, db_1$$

subject to

$$\ell_1 \geq (1-\lambda)\tilde{F}_1^{-1}(1-\beta_1) - K_1,$$
$$\ell_1 \geq 0,$$
$$D - \tilde{F}_2\left\{\frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]y_m}{1-\lambda}\right\} \geq 0,$$
$$D - \tilde{F}_2\left\{\frac{0.827\ell_1 + K_2 - [1-\lambda(1+i)]\tilde{F}_1^{-1}(1-\beta_2)}{1-\lambda}\right\} \geq 0, \tag{33}$$
$$y_m \geq \tilde{F}_1^{-1}(1-\beta_2),$$
$$y_m \leq h,$$
$$0 \leq D \leq 1.$$

By solving Eq 33 and then putting ℓ_1', D'', and y_m' into Eq 12 and using Eq 32 the optimal $\ell_2'(b_1)$ is derived. Hence Eq 9 has been solved.

Conclusions

It has been shown in the preceding section that when the constraint $P(\ell_2 \geq 0) \geq \beta_2$ is not binding the result is

$$\ell_2' = [1 - \lambda(1+i)]\, b_1 - 0.827\, \ell_1' - K_2 + (1-\lambda)\, \tilde{F}_1 (1-\beta_2) \text{ for } g \le b_1 \le h .$$

In other words, $\ell_2' (b_1)$ is a linear function of b_1 and as such it coincides with the optimal linear rule found by Charnes and Thore[5] for this same case. It is emphasized again that in finding the optimal linear rule Charnes and Thore[5] required b_1 and b_2 to be normal random variables, although the results hold for a much more general class of random variables.

In order to give a brief interpretation of these linear rules write

$$\ell_1' = \gamma_1$$

$$\ell_2' = \gamma_{21} (b_1 - E_1) + \gamma_2 ,$$

where $\gamma_1, \gamma_2, \gamma_{21}$ are constants and E_1 is the mean of the random variable b_1 (i.e., ΔS_1).

Since $\gamma_{21} \equiv 1 - \lambda\,(1+i) > 0$, the optimal decision rule ℓ_2' states that if there has been an inflow of savings during period 1 that exceeds the mean, free cash holdings during period 2 will be lowered by an amount $\gamma_{21} (b_1 - E_1)$, as opposed to the case in which ℓ_2' is chosen to be a zero-order rule (i.e., $\ell_2' = \gamma_2$).

It is also clear from the solution that the holdings of free cash will be equal to their minimum values as long as these minimum values do not interfere with the limitations of long-term borrowing given by the $P(\ell_t \geq 0) \geq \beta_t$, $t = 1, 2$, constraints. When these constraints become binding the amount of free cash held will be greater than this minimum value. Thus the association's profits drop, owing to the loss in interest that would be obtained if less money were held and, consequently, more money loaned.

It is worth noting at this point that if Eq 5 were expanded to be a full n dimensional problem and if it were assumed that the constraints $P(\ell_j \geq 0) \geq \beta_j$ $j = 1, \ldots, n$, are not binding then by theorem 6 in Charnes and Kirby[6] it can be seen that the optimal rule for ℓ_j will, in fact, be the optimal linear rule. This result certainly gives great justification for using linear decision rules in cases where the computation of the optimal decision rule from a much larger class of decision rules is extremely difficult to do.

Consider the case where the constraint $P(\ell_2 \geq 0) \geq \beta_2$ is binding. The first observation is that in this case ℓ_2' is discontinuous. The fact that it must have at least one discontinuity follows directly from Eq 32, since it is known that $y_m' > \tilde{F}_1^{-1} (1 - \beta_2)$ when $P(\ell_2 \geq 0) \geq \beta_2$ is binding. In general, however, ℓ_2' will also be discontinuous at $b_1 = \tilde{F}_1^{-1}(1 - \beta_2)$.

$\ell_2' (b_1)$ will be continuous at $b_1 = \tilde{F}_1^{-1} (1 - \beta_2)$ if and only if $\ell_2' [\tilde{F}_1^{-1} (1 - \beta_2)] = 0$, i.e., if and only if $[1 - \lambda\,(1+i)]\, \tilde{F}_1^{-1} (1-\beta_2) - 0.827\, \ell_1' - K_2 + (1 - \lambda)\, \tilde{F}_1^{-1} (D^*) = 0$.

One way of understanding why ℓ_2' is discontinuous is by the following intuitive argument: suppose that when $\beta_2 = \beta_2^0$ the optimal rule is linear, but it is discontinuous for all $\beta_2 > \beta_2^0$. Suppose that β_2 is increased to $\beta_2 = \beta_2^0 + \epsilon$. It is known then that to get ℓ_2' it is necessary to increase the measure of the

set of points for which $\ell_2^0 > 0$ where ℓ_2^0 is the optimal rule when $\beta_2 = \beta_2^0$, i.e., ℓ_2^0 must be increased for some values of b_1 to get ℓ_2'. The $1-\alpha_2$ constraint being satisfied by ℓ_2^0 as an equality means that ℓ_2^0 must also decrease for some values of b_1, otherwise this constraint would be violated. The question then arises, for what values of b_1 should ℓ_2^0 be increased and for what values should ℓ_2^0 be decreased in order to get ℓ_2'. It seems reasonable that ℓ_2' should be as close to ℓ_2^0 as possible. Therefore ℓ_2^0 should be increased for those values of b_1 for which $\ell_2^0 < 0$ but is almost 0. Similarly ℓ_2^0 should be decreased where it is positive but almost 0. This accounts for the location of the region $\ell_2' = 0$. For all remaining values of b_1, ℓ_2^0 is changed as little as possible by making ℓ_2' linear with the same slope as ℓ_2^0 and changing the intercept term just enough to satisfy the $1-\alpha_2$ constraint.

If it is assumed that b_1 and b_2 are $N(E_t, \delta_t)$, $t = 1, 2$, random variables, it can be shown that the optimal linear rule when the β_2 constraint is binding is

$$\ell_2' = \left(1-\lambda(1+i) - \left\{\frac{(1-\lambda)\,\delta_2\,(0.173)\,\Phi^{-1}(1-\beta_2)\,\delta_1}{\sqrt{[\Phi^{-1}(1-\alpha_2)]^2 - [0.173\,\Phi^{-1}(1-\beta_2)\,\delta_1]^2}}\right\}\right)(b_1 - E_1)$$
$$+\ \Phi^{-1}(1-\beta_2)\,\delta_1\left|1-\lambda(1+i) - \left\{\frac{(1-\lambda)\,\delta_2\,(0.173)\,\Phi^{-1}(1-\beta_2)\,\delta_1}{\sqrt{[\Phi^{-1}(1-\alpha_2)]^2 - [0.173\,\Phi^{-1}(1-\beta_2)\,\delta_1]^2}}\right\}\right|, \quad (34)$$

where Φ is the distribution function of a N(0, 1) random variable.

Now suppose that $\beta_2 > \frac{1}{2}$, then $\Phi^{-1}(1-\beta_2) < 0$, so that the slope of ℓ_2' in Eq 34 is greater than the slope of ℓ_2' in Eq 32. This means that the optimal linear rule would recommend borrowing more when b_1 is very negative and lending more when b_1 is very positive than would the optimal rule given by Eq 32.

Conversely if $\beta < \frac{1}{2}$, the slope of Eq 34 is less than the slope of Eq 32 when $\ell_2' \neq 0$. In this case using the linear rule recommends borrowing less and lending less for small and large values of b_1 respectively, than would Eq 32.

In conclusion, there is one further reason for not being too surprised at the discontinuity of ℓ_2' under certain circumstances. Equation 10 can be regarded as a problem for finding the "optimal control" $\ell_2(b_1)$.[8] From control theory it is known that in many cases the optimal control is discontinuous. It takes on one value for certain time intervals and then switches abruptly to another value. This is precisely what ℓ_2' does. It is defined by Eq 32 for all values of b_1 in $[g, \tilde{F}_1^{-1}(1-\beta_2)]$, it then switches to $\ell_2' = 0$ from $\tilde{F}_1^{-1}(1-\beta_2)$ to y_m', and then switches again to the line defined by Eq 32. Thus ℓ_2' behaves similarly to the optimal controls in many control-theory problems.

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